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Error Bounds for Smooth Interpolation in Triangles

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1. INTRODUCTION

The main purpose of this paper is to present improved error bounds for the smooth rational interpolation schemes over triangles discussed in Barnhill, Birkhoff, and Gordon [1]. This is accomplished by an appropriate extension of the Sard kernel theorem to triangles, a second purpose of this paper being to show that the Sard kernel theorem can be used on more general regions than rectangles.

In Ref. 1, an interpolation scheme using rational functions is described which interpolates to a given function and its first $k - 1$ derivatives on the boundary of a triangle T. In the case $k = 1$, or the trilinear case, the authors prove that if $u \in C^4(T)$ the error in interpolation is $O(h^3)$ where h is the diameter of T. For the case $k = 2$, or the tricubic case, they state, but do not prove that the error in interpolation is $O(h^6)$ for functions $u \in C^9(T)$.

In this paper we show that one obtains $O(h^3)$ in the trilinear case assuming only that u is contained in a certain Sard-like space $B_{1,2}^{\infty}$, a slightly weaker assumption than $u \in C^3(T)$. In the general case we show that the error in interpolation is $O(h^{3k})$ assuming that $u \in B_{s,3k-s}^{\infty}$, where $s = [3k/2]$, a slightly weaker assumption than $u \in C^{3k}(T)$. Moreover, we show how to calculate explicit bounds for the constants involved in these error estimates, and carry out this calculation for the trilinear case. We remark that the determination

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that the order of the error is $3k$ can be obtained from Theorem 5 of Ciarlet and Raviart [6] for u belonging to the Sobolev space $W_\infty^{3k}(T)$ once one shows that the interpolation scheme reproduces all polynomials of degree $3k - 1$ or less. Our proof allows one, in principal, to compute the constants involved.

The interpolation scheme described in Ref. 1 has as one application that it can be used to generate compatible triangular finite elements by taking the boundary data to be polynomials of certain degrees. This is carried out in Ref. 5. A second type of interpolant is introduced in Ref. 3. On one side of the triangle, one could instead take the boundary data to be arbitrary functions so as to incorporate finite elements over triangulated polygons which satisfy boundary conditions exactly.

For algebraic simplicity, the authors of Ref. 1 considered a "standard" triangle with vertices at (0, 0), (1, 0), and (1, 1), noting that any other triangle can be obtained from it by an affine transformation which carries polynomial and rational functions into polynomial and rational functions of the same degree and preserves the order of approximation. We shall also find it convenient to consider a "standard" triangle but choose to consider the triangle T_h with vertices at $(0, 0)$, $(h, 0)$, and $(0, h)$. Our methods, however, are applicable to any triangle.

2. The Sard Kernel Theorem in Triangles

In this section we describe the Sard kernel theorem as it relates to triangles, in preparation for later sections. Our discussion parallels that in Barnhill and Gregory [2]. We let T be any triangle with its longest side parallel to a coordinate axis and let (a, b) be any point in T such that the rectangle with opposite corners (a, b) and (x, y) and with sides parallel to the coordinate axes is contained in T for all (x, y) in T. For example, one can choose (a, b) to be the point on the longest side which is the foot of the perpendicular to this side from the opposite vertex. In the case of a right triangle one can also choose (a, b) to be the vertex at the right angle.

For all $u \in C^m(T)$, one has the Taylor expansion

$$\begin{aligned} u(x, y) = & \sum_{i+j<m} u_{i,j}(a, b)(x-a)^{(i)}(y-b)^{(j)} \\ & + \sum_{j<q} (y-b)^{(j)} \int_a^x (x-\tilde{x})^{(m-j-1)}\, u_{m-j,j}(\tilde{x}, y)\, d\tilde{x} \\ & + \sum_{i<p} (x-a)^{(i)} \int_b^y (y-\tilde{y})^{(m-i-1)}\, u_{i,m-i}(x, \tilde{y})\, d\tilde{y} \\ & + \int_a^x \int_b^y (x-\tilde{x})^{(p-1)}(y-\tilde{y})^{(q-1)}\, u_{p,q}(\tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y}, \quad p+q=m. \end{aligned} \tag{2.1}$$

Here we use subscripts to denote partial derivatives and use the notation

$$(x-a)^{(i)} = \frac{(x-a)^i}{i!}.$$

Actually (2.1) holds for the wider class of functions $B_{p,q}^1 = B_{p,q}^1(a, b)$ with the properties

$$\begin{aligned} & u_{i,j} \in C(T), \qquad i < p, \qquad j < q, \\ & u_{m-j-1,j}(x, b) \text{ is abs. cont.}, \; u_{m-j,j}(x, b) \in L_1(I_1), \qquad 0 \leqslant j < q \\ & u_{i,m-i-1}(a, y) \text{ is abs. cont.}, \; u_{i,m-i}(a, y) \in L_1(I_2), \qquad 0 \leqslant i < p \\ & u_{p,q} \in L_1(T), \end{aligned} \tag{2.2}$$

where I_1 is the intersection of the line $y = b$ with T and I_2 is the intersection of the line $x = a$ with T. Since we will want to use certain Hölder inequalities later, we will want to restrict our class of functions to be the class $B_{p,q}^r = B_{p,q}^r(a, b)$, $r \geqslant 1$, with the properties (2.2) but with

$$\begin{aligned} u_{m-j,j}(x, b) &\in L_r(I_1), \qquad 0 \leqslant j < q, \\ u_{i,m-i}(a, y) &\in L_r(I_2), \qquad 0 \leqslant i < p, \\ u_{p,q} &\in L_r(T). \end{aligned} \tag{2.3}$$

The class $B_{p,q}^1$ is similar to the class of functions boldface $\mathbf{B}_{p,q}$ over rectangles in Sard [8]. Also $B_{p,q}^2$ is similar to the class of functions $T^{p,q}(a, b)$ discussed in Mansfield [7].

Let F be a linear functional of the form

$$\begin{aligned} F(u) = & \sum_{\substack{i<p \\ j<q}} \iint_T u_{i,j}(x, y)\, d\mu^{i,j}(x, y) + \sum_{\substack{i+j<m \\ i \geqslant p}} \int_{I_1} u_{i,j}(x, b)\, d\mu^{i,j}(x) \\ & + \sum_{\substack{i+j<m \\ j \geqslant q}} \int_{I_2} u_{i,j}(a, y)\, d\mu^{i,j}(y), \end{aligned} \tag{2.4}$$

where the functions $\mu^{i,j}$ are of bounded variation. Suppose also that F has the property that $F(q) = 0$ for all polynomials q of degree $m - 1$ or less. Then the Sard Kernel theorem [8, p. 175] can be applied to give the representation

$$\begin{aligned} F(u) = & \sum_{j<q} \int_{I_1} K^{m-j,j}(x, y; \tilde{x})\, u_{m-j,j}(\tilde{x}, b)\, d\tilde{x} \\ & + \sum_{i<p} \int_{I_2} K^{i,m-i}(x, y; \tilde{y})\, u_{i,m-i}(a, \tilde{y})\, d\tilde{y} \\ & + \iint_T K^{p,q}(x, y; \tilde{x}, \tilde{y})\, u_{p,q}(\tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y}, \qquad u \in B_{p,q}^1(a, b), \end{aligned} \tag{2.5}$$

where

$$K^{m-j,j}(x, y; \tilde{x}) = F_{(x,y)}[(x-\tilde{x})^{(m-j-1)}\,\psi(a, \tilde{x}, x)(y-b)^{(j)}], \quad j < q, \tag{2.6}$$

$$K^{i,m-i}(x, y; \tilde{y}) = F_{(x,y)}[(x-a)^{(i)}(y-\tilde{y})^{(m-i-1)}\,\psi(b, \tilde{y}, y)], \quad i < p, \tag{2.7}$$

$$K^{p,q}(x, y; \tilde{x}, \tilde{y}) = F_{(x,y)}[(x-\tilde{x})^{(p-1)}\,\psi(a, \tilde{x}, x)(y-\tilde{y})^{(q-1)}\,\psi(b, \tilde{y}, y)], \tag{2.8}$$

where

$$\psi(a, \tilde{x}, x) = \begin{cases} 1 & \text{if } a \leqslant \tilde{x} < x \\ -1 & \text{if } x \leqslant \tilde{x} < a \\ 0 & \text{otherwise.} \end{cases}$$

3. Trilinear Interpolation in Triangles

In this section we derive a priori error bounds including constants for the trilinear interpolant of Ref. 1. The trilinear interpolant on the triangle T_h with vertices (0, 0), $(h, 0)$, and $(0, h)$ is given by Q^*u where

$$Q^* = \tfrac{1}{2}[\mathscr{P}_1 + \mathscr{P}_2 + \mathscr{P}_3 - \mathscr{L}] \tag{3.1}$$

where

$$\mathscr{P}_1 u(x, y) = \left(\frac{h-x-y}{h-y}\right) u(0, y) + \left(\frac{x}{h-y}\right) u(h-y, y), \tag{3.2}$$

$$\mathscr{P}_2 u(x, y) = \left(\frac{h-x-y}{h-x}\right) u(x, 0) + \left(\frac{y}{h-x}\right) u(x, h-x), \tag{3.3}$$

$$\mathscr{P}_3 u(x, y) = \left(\frac{x}{x+y}\right) u(x+y, 0) + \left(\frac{y}{x+y}\right) u(0, x+y), \tag{3.4}$$

$$\mathscr{L} u(x, y) = \left(\frac{h-x-y}{h}\right) u(0, 0) + \frac{x}{h} u(h, 0) + \frac{y}{h} u(0, h), \tag{3.5}$$

where the projector $\mathscr{L} = \mathscr{P}_i \mathscr{P}_j \mathscr{P}_k$, $i \neq j$, $j \neq k$, $k \neq i$, the product being taken in any order. It is shown in Ref. 1 that Q^*u interpolates u on the boundary of T_h and also that Q^* reproduces all quadratics but not all cubics.

Theorem 2.1. *If* $u \in B_{1,2}^{\infty}(0, 0)$, *then*

$$\|(I - Q^*)\, u(x, y)\|_{L_\infty(T_h)} \leqslant \frac{h^3}{8}\,[\|u_{3,0}(\tilde{x}, 0)\|_{L_\infty[0,h]} + \|u_{2,1}(\tilde{x}, 0)\|_{L_\infty[0,h]} + \|u_{0,3}(0, \tilde{y})\|_{L_\infty[0,h]}] + \frac{h^3}{4}\,\|u_{1,2}(\tilde{x}, \tilde{y})\|_{L_\infty(T_h)}\,. \tag{3.6}$$

If $u \in B_{1,2}^{2}(0,0)$, *then*

$$\|(I-Q^*)\,u(x,y)\|_{L_2(T_h)} \leqslant \frac{h^{7/2}}{12\sqrt{5}}\,[\|u_{3,0}(\tilde{x},0)\|_{L_2[0,h]} + \|u_{2,1}(\tilde{x},0)\|_{L_2[0,h]} + \|u_{0,3}(0,\tilde{y})\|_{L_2[0,h]}] + \frac{h^3}{4\sqrt{2}}\,\|u_{1,2}(\tilde{x},\tilde{y})\|_{L_2(T_h)}\,. \tag{3.7}$$

In general, if $u \in B_{1,2}^{r}(0,0)$, *then*

$$\|(I-Q^*)\,u(x,y)\|_{L_r(T_h)} \leqslant \frac{h^{3+1/r}[B(r+1,r+2)]^{1/r}}{2(r+1)^{1/r}} \times [\|u_{3,0}(\tilde{x},0)\|_{L_r[0,h]} + \|u_{2,1}(\tilde{x},0)\|_{L_r[0,h]} + \|u_{0,3}(0,\tilde{y})\|_{L_r[0,h]}] + \frac{h^3}{2^{2-1/r}(r+2)^{1/r}}\,\|u_{1,2}(\tilde{x},\tilde{y})\|_{L_r(T_h)}\,, \tag{3.8}$$

where $B(\mu,\eta)$ *is the beta function.*

Proof. We use (2.5) for $u \in B_{1,2}^{r}(0,0)$ with $F = I - Q^*$. First

$$K^{3,0}(x,y;\tilde{x}) = (I-Q^*)(x-\tilde{x})_+^{(2)} = \frac{1}{2}\left\{(x-\tilde{x})_+^{(2)} - \left(\frac{x}{h-y}\right)(h-y-\tilde{x})_+^{(2)} - \left(\frac{x}{x+y}\right)(x+y-\tilde{x})_+^{(2)} + \frac{x}{h}(h-\tilde{x})^{(2)}\right\}.$$

Then $H(\alpha) \equiv G(\alpha,\tilde{x}) = \frac{1}{2}(x/\alpha)(\alpha-\tilde{x})_+^{(2)}$ is a monotonically increasing function of α for $0 \leqslant \alpha \leqslant h$, so that $H(h) - H(h-y) \geqslant 0$ and $H(x) - H(x+y) \leqslant 0$. Also, $G(\alpha,\tilde{x})$ is a monotonically decreasing function of $\tilde{x}$ for $0 \leqslant \tilde{x} \leqslant h$ so that $\max_{0\leqslant\tilde{x}\leqslant h}[G(h,\tilde{x}) - G(h-y,\tilde{x})]$ occurs at $\tilde{x}=0$ and equals $xy/4$. Similarly $\min_{0\leqslant\tilde{x}\leqslant h}[H(x) - H(x+y)]$ occurs at $\tilde{x}=0$ and equals $-xy/4$. Hence

$$|K^{3,0}(x,y;\tilde{x})| \leqslant xy/2 \tag{3.9}$$

uniformly in $\tilde{x}$. We do not claim that this is a sharp result, but we do note that $K^{3,0}(x,y;\tilde{x})$ is not a monotonic function of $\tilde{x}$. Next

$$K^{2,1}(x,y;\tilde{x}) = (I-Q^*)[(x-\tilde{x})_+\,y] = \frac{1}{2}\left\{(x-\tilde{x})_+\,y - \left(\frac{xy}{h-y}\right)(h-y-\tilde{x})_+\right\} \leqslant 0,$$

and

$$\min_{0 \leqslant \tilde{x} \leqslant h} K^{2,1}(x, y; \tilde{x}) = K^{2,1}(x, y; x) = -\frac{xy(h-y-x)}{2(h-y)}. \tag{3.10}$$

$$\begin{aligned} K^{1,2}(x, y; \tilde{x}, \tilde{y}) &= (I - Q^*)[(x-\tilde{x})_+^0 (y-\tilde{y})_+] \\ &= \frac{1}{2}\Big\{(y-\tilde{y})_+ \Big[(x-\tilde{x})_+^0 - \Big(\frac{x}{h-y}\Big)(h-y-\tilde{x})_+^0\Big] \\ &\quad + (x-\tilde{x})_+^0 \Big[(y-\tilde{y})_+ - \Big(\frac{y}{h-x}\Big)(h-x-\tilde{y})_+\Big]\Big\}. \end{aligned}$$

Since

$$\Big| (x-\tilde{x})_+^0 - \Big(\frac{x}{h-y}\Big)(h-y-\tilde{x})_+^0 \Big| \leqslant 1$$

and

$$\Big| (y-\tilde{y})_+ - \Big(\frac{y}{h-x}\Big)(h-x-\tilde{y}) \Big| \leqslant \Big(\frac{y}{h-x}\Big)(h-x-y),$$

then

$$| K^{1,2}(x, y; \tilde{x}, \tilde{y})| \leqslant \tfrac{1}{2}(h-x). \tag{3.11}$$

Finally $K^{0,3}(x, y; \tilde{y})$ is analogous to $K^{3,0}(x, y; \tilde{x})$ so that

$$| K^{0,3}(x, y; \tilde{y})| \leqslant xy/2 \tag{3.12}$$

uniformly in $\tilde{y}$.

Equation (2.5) and the triangle inequality imply that

$$\begin{aligned} &|(I - Q^*)\, u(x, y)| \\ &\quad \leqslant \int_0^h | K^{3,0}(x, y; \tilde{x})\, u_{3,0}(\tilde{x}, 0)|\, d\tilde{x} \\ &\qquad + \int_0^h | K^{2,1}(x, y; \tilde{x})\, u_{2,1}(\tilde{x}, 0)|\, d\tilde{x} \\ &\qquad + \iint_{T_h} | K^{1,2}(x, y; \tilde{x}, \tilde{y})\, u_{1,2}(\tilde{x}, \tilde{y})|\, d\tilde{x}\, d\tilde{y} \\ &\qquad + \int_0^h | K^{0,3}(x, y; \tilde{y})\, u_{0,3}(0, \tilde{y})|\, d\tilde{y} \\ &\quad \leqslant \frac{xy}{2}\Big[\int_0^h | u_{3,0}(\tilde{x}, 0)|\, d\tilde{x} + \int_0^h | u_{2,1}(\tilde{x}, 0)|\, d\tilde{x} + \int_0^h | u_{0,3}(0, \tilde{y})|\, d\tilde{y}\Big] \\ &\qquad + \frac{h-x}{2} \iint_{T_h} | u_{1,2}(\tilde{x}, \tilde{y})|\, d\tilde{x}\, d\tilde{y}, \end{aligned}$$

from which (3.6) and (3.7) follow. In order to obtain (3.8) we observe that the integral $\int_0^h y^r(h-y)^{r+1}\,dy$, after a change of variable to [0, 1] becomes the beta function $B(r+1, r+2)$. If r is an integer, then $B(r+1, r+2) = [r!\,(r+1)!]/[(2r+2)!]$.

4. Higher Order Interpolation

We now give error bounds for interpolation by the projector

$$Q^* = \tfrac{1}{2}[\mathscr{P}_1^k + \mathscr{P}_2^k + \mathscr{P}_3^k - \mathscr{L}^k] \tag{4.1}$$

where $\mathscr{L}^k = \mathscr{P}_i^k\mathscr{P}_j^k\mathscr{P}_l^k$, $i \neq j \neq l$, the product being taken in any order. Here the projectors $\mathscr{P}_i^k$ are the Hermite interpolants to values and the first $k-1$ directional derivatives (parallel to the ith side) on two sides of the triangle T_h interpolated along parallels to the third side. Explicitly

$$\mathscr{P}_1^k u(x,y) = \sum_{i<k} (h-y)^i H_{1,i}\left(\frac{x}{h-y}\right) u_{i,0}(0,y) + \sum_{i<k} (h-y)^i H_{2,i}\left(\frac{x}{h-y}\right) u_{i,0}(h-y,y), \tag{4.2}$$

$$\mathscr{P}_2^k u(x,y) = \sum_{i<k} (h-x)^i H_{1,i}\left(\frac{y}{h-x}\right) u_{0,i}(x,0) + \sum_{i<k} (h-x)^i H_{2,i}\left(\frac{y}{h-x}\right) u_{0,i}(x,h-x), \tag{4.3}$$

$$\mathscr{P}_3^k u(x,y) = \sum_{i<k} (x+y)^i H_{1,i}\left(\frac{x}{x+y}\right)\left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i u\right)(0,x+y) + \sum_{i<k} (x+y)^i H_{2,i}\left(\frac{x}{x+y}\right)\left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i u\right)(x+y,0), \tag{4.4}$$

where the $H_{j,i}$, $j = 1, 2$; $i = 0,\ldots, k-1$, are the cardinal Hermite polynomials of degree $2k-1$ on [0, 1]. We now show that $\mathscr{L}^k u$ is an element of $\mathscr{T}_{2k-1}$, the set of all polynomials which are of degree $2k-1$ on all parallels to each of the lines $x = 0$, $y = 0$, $h - x - y = 0$.

Lemma 4.1. *The product $\mathscr{P}_i^k\mathscr{P}_j^k\mathscr{P}_l^k$, $i \neq j \neq l$, in any order of the three projectors $\mathscr{P}_1^k$, $\mathscr{P}_2^k$, $\mathscr{P}_3^k$ is a projector with range $\mathscr{F}_{2k-1}$.*

Proof. Since $\mathscr{T}_{2k-1}$ is invariant under any affine transformation that maps the set of vertices of T_h onto themselves, we need only show that $\mathscr{P}_1{}^k\mathscr{P}_3{}^k\mathscr{P}_2{}^k$ is a projector with range $\mathscr{T}_{2k-1}$. First $\mathscr{P}_3{}^k\mathscr{P}_2{}^k u$ may be expressed as

$$\begin{aligned}\mathscr{P}_3{}^k\mathscr{P}_2{}^k u(x, y) = {} & \sum_{i,j<k} \rho^{(i,j)}(x, y)\, u_{i,j}(0, 0) \\ & + \sum_{i,j<k} \sigma^{(i,j)}(x, y) \left(\left[\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^i \frac{\partial^j}{\partial y^j}\right] u\right)(0, h) \\ & + \sum_{i<k} \tau^{(i)}(x, y) \left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i u\right)(x + y, 0) \end{aligned} \tag{4.5}$$

where the $\rho^{(i,j)}$, $\sigma^{(i,j)}$, and $\tau^{(i)}$ are rational functions. The first two terms come from evaluating

$$\sum_{i<k} (x + y)^i H_{1,i}\left(\frac{x}{x + y}\right)\left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i \mathscr{P}_2{}^k u\right)(0, x + y).$$

To evaluate

$$\sum_{i<k} (x + y)^i H_{2,i}\left(\frac{x}{x + y}\right)\left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i \mathscr{P}_2{}^k u\right)(x + y, 0) \tag{4.6}$$

we use the fact that $H_{1,i}^{(j)}(0) = \delta_{ij}$, $0 \leqslant i, j \leqslant k - 1$ and $H_{2,i}^{(j)}(0) = 0$, $0 \leqslant i$, $j \leqslant k - 1$. We also observe that

$$\frac{\partial^i}{\partial x^i}\left[(h - x)^j H_{1,j}\left(\frac{y}{h - x}\right)\right](x + y, 0) = 0,$$

and

$$\frac{\partial^i}{\partial y^i}\left[(h - x)^j H_{1,j}\left(\frac{y}{h - x}\right)\right](x + y, 0) = \delta_{ij}, \quad 0 \leqslant i, \quad j \leqslant k - 1.$$

Thus

$$\frac{\partial^{\mu+n}}{\partial x^\mu\, \partial y^n}\left[\sum_{j<k} (h - x)^j H_{1,j}\left(\frac{y}{h - x}\right) u_{0,j}(x, 0)\right] = \delta_{nj} u_{\mu,j}(x + y, 0)$$

and (4.6) reduces to the third term of (4.5). Then

$$\begin{aligned}\mathscr{P}_1{}^k\mathscr{P}_3{}^k\mathscr{P}_2{}^k u(x, y) = {} & \sum_{i,j<k} \alpha^{(i,j)}(x, y)\, u_{i,j}(0, 0) \\ & + \sum_{i,j<k} \beta^{(i,j)}(x, y) \left(\left[\frac{\partial^i}{\partial x^i}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^j\right] u\right)(h, 0) \\ & + \sum_{i,j<k} \gamma^{(i,j)}(x, y) \left(\left[\frac{\partial^i}{\partial y^i}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^j\right] u\right)(0, h) \end{aligned} \tag{4.7}$$

where the $\alpha^{(i,j)}$, $\beta^{(i,j)}$, $\gamma^{(i,j)}$ are apparently rational functions. We now show that these functions are actually all elements of $\mathscr{T}_{2k-1}$. From the definition of the $\mathscr{P}_i^k$, the range of $\mathscr{P}_1^k\mathscr{P}_3^k\mathscr{P}_2^k$ certainly contains $\mathscr{T}_{2k-1}$. Also the number of elements in $\mathscr{T}_{2k-1}$ is exactly the same as the number of terms in (4.7). It then follows that the interpolation conditions of (4.7) are linearly independent over $\mathscr{T}_{2k-1}$ and the $\alpha^{(i,j)}$, $\beta^{(i,j)}$, $\gamma^{(i,j)}$ are all elements of $\mathscr{T}_{2k-1}$. They are, in fact, the cardinal functions for the interpolation scheme (4.7). Q.E.D.

In what follows it will be convenient to express $\mathscr{L}^k u(x, y)$ as

$$\begin{aligned}\mathscr{L}^k u(x, y) = &\sum_{i,j<k} h^{i+j}\alpha^{(i,j)}\left(\frac{x}{h}, \frac{y}{h}\right) u_{i,j}(0, 0)\\ &+ \sum_{i,j<k} h^{i+j}\beta^{(i,j)}\left(\frac{x}{h}, \frac{y}{h}\right)\left(\left[\frac{\partial^i}{\partial x^i}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^j\right] u\right)(h, 0)\\ &+ \sum_{i,j<k} h^{i+j}\gamma^{(i,j)}\left(\frac{x}{h}, \frac{y}{h}\right)\left(\left[\frac{\partial}{\partial y^i}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^j\right] u\right)(0, h)\end{aligned}$$

where the $\alpha^{(i,j)}$, $\beta^{(i,j)}$, $\gamma^{(i,j)}$ are the cardinal functions for the corresponding interpolation scheme on the triangle T_1 with vertices (0, 0), (1, 0), (0, 1). We remark that for the case $k = 2$, $\mathscr{L}^k u$ is the tricubic polynomial interpolant constructed by Birkhoff [4].

We now show that Q^*u interpolates u and its first $k - 1$ derivatives on the boundary of T_h. This will be implied by the following two lemmas.

LEMMA 4.2. *For $i, j = 1, 2, 3$, $i \neq j$, the functions*

$$(\mathscr{P}_i^k \oplus \mathscr{P}_j^k)u = (\mathscr{P}_i^k + \mathscr{P}_j^k - \mathscr{P}_i^k\mathscr{P}_j^k)u \quad \textit{interpolate} \quad u \in C^{k-1}(\partial T_h)$$

and its first $k - 1$ normal derivatives on ∂T_h provided that u satisfies the compatibility conditions

$$\left(\frac{\partial^{m+n}u}{\partial s_i^m\, \partial s_j^n}\right)(P_l) = \left(\frac{\partial^{m+n}u}{\partial s_j^n\, \partial s_i^m}\right)(P_l), \quad m, n < k; \quad m + n > k - 1 \tag{4.8}$$

at the vertex P_l with adjacent sides i and j, where $\partial/\partial s_i$ denotes directional differentiation parallel to the ith side.

Remark. For the case $k = 2$, this is essentially a restatement of Lemma 4 of Ref. 1, except that the assumption $u \in C^2(T)$ in Ref. 1 is stronger than necessary.

Proof of Lemma 4.2. By affine invariance and symmetry, it is sufficient to consider the case $(\mathscr{P}_1^k \oplus \mathscr{P}_2^k)$. First, it is clear that

$$u - (\mathscr{P}_1^k \oplus \mathscr{P}_2^k)u = (I - \mathscr{P}_1^k)(I - \mathscr{P}_2^k)u = (I - \mathscr{P}_1^k)[(I - \mathscr{P}_2^k)u]$$

is zero and has all its partial derivatives of order less than or equal to $k-1$ zero on the sides $x = 0$ and $h - x - y = 0$. We now express $(\mathscr{P}_1{}^k \oplus \mathscr{P}_2{}^k)u$ as

$$(\mathscr{P}_1{}^k \oplus \mathscr{P}_2{}^k)u = [\mathscr{P}_2{}^k + \mathscr{P}_1{}^k(I - \mathscr{P}_2{}^k)]u.$$

Since $(I - \mathscr{P}_2{}^k)\, u(x, 0) = 0$, $\mathscr{P}_1{}^k v$ with $v = (I - \mathscr{P}_2{}^k)u$ is zero on $y = 0$ and thus $(\mathscr{P}_1{}^k \oplus \mathscr{P}_2{}^k)\, u(x, 0) = u(x, 0)$. Now

$$\left(\frac{\partial^m \mathscr{P}_1{}^k v}{\partial y^m}\right)(x, y) = \sum_{i=0}^{k-1} \sum_{j=0}^{m} \binom{m}{j} \psi_{i,j}^{(1)}(x, y) \frac{\partial^{m-j}}{\partial y^{m-j}} \left(\frac{\partial^i u}{\partial x^i}(0, y)\right)$$
$$+ \sum_{i=0}^{k-1} \sum_{j=0}^{m} \binom{m}{j} \psi_{i,j}^{(2)}(x, y) \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)^{m-j} \left(\frac{\partial^i u}{\partial x^i}(h - y, y)\right),$$
$$1 \leqslant m < k, \quad (4.9)$$

where

$$\psi_{i,j}^{(\mu)}(x, y) = \partial^j \left[(h - y)\, H_{\mu,i}\left(\frac{x}{h - y}\right)\right] \Big/ \partial y^j, \qquad \mu = 1, 2.$$

We now let $v = (I - \mathscr{P}_2{}^k)u$ and evaluate (4.9) at $y = 0$. Because of our continuity assumption on u and the fact that v and all its partial derivatives of order less than or equal to $k - 1$ are zero on $x = 0$ and $h - x - y = 0$,

$$\frac{\partial^j}{\partial y^j}\left(\frac{\partial^i v}{\partial x^i}\right)(0, 0) = 0,$$

and

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)^j \left(\frac{\partial^i v}{\partial x^i}\right)(h, 0) = 0, \qquad i + j < k.$$

Also

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)^j \left(\frac{\partial^i v}{\partial x^i}\right)(h - y, y) = 0, \qquad 0 \leqslant j \leqslant m, \qquad 0 \leqslant i \leqslant k - 1,$$

since the last j derivatives are taken along the line $h - x - y = 0$ and $\partial^i(I - \mathscr{P}_2{}^k)u/\partial x^i \equiv 0$ there, $0 \leqslant i \leqslant k - 1$.

Thus

$$\left(\frac{\partial^m \mathscr{P}_1{}^k v}{\partial y^m}\right)(x, 0) = \sum_{j=0}^{m-1} \sum_{i=1}^{m-j} \binom{m}{j} \psi_{k-i,j}^{(1)}(x, 0) \left(\frac{\partial^{m-j}}{\partial y^{m-j}} \left(\frac{\partial^{k-i} v}{\partial x^{k-i}}\right)\right)(0, 0).$$
$$(4.10)$$

By direct calculation, using the formula for $\partial^{k-i}\mathscr{P}_2{}^k u/\partial x^{k-i}$ which can be obtained from (4.9) by interchanging x and y, one finds that

$$\left(\frac{\partial^{m-j}}{\partial y^{m-j}}\left(\frac{\partial^{k-i}\mathscr{P}_2{}^k u}{\partial x^{k-i}}\right)\right)(0,0) = \left(\frac{\partial^{k-i}}{\partial x^{k-i}}\left(\frac{\partial^{m-j} u}{\partial y^{m-j}}\right)\right)(0,0). \qquad (4.11)$$

Hence

$$\left(\frac{\partial^m}{\partial y^m}\left[\mathscr{P}_2{}^k + \mathscr{P}_1{}^k(I - \mathscr{P}_2{}^k)u\right]\right)(x,0) = \frac{\partial^m u}{\partial y^m}(x,0), \qquad 1 \leqslant m \leqslant k-1 \qquad (4.12)$$

if and only if

$$\frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial x^n}u(0,0) = \frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}u(0,0), \qquad m, n < k, \qquad m + n > k - 1, \qquad (4.13)$$

which proves the lemma.

LEMMA 4.3. *If $u \in C^{k-1}(\partial T_h)$ and satisfies* (4.8) *at each vertex, then Q^*u, where Q^* is defined by* (4.1) *can be expressed as*

$$Q^*u = W_i u = \tfrac{1}{2}\{(\mathscr{P}_i{}^k \oplus \mathscr{P}_j{}^k) + (\mathscr{P}_i{}^k \oplus \mathscr{P}_l{}^k)\}u, \quad i \neq j, \quad j \neq l, \quad l \neq i, \quad i = 1, 2, 3. \qquad (4.14)$$

Proof.

$$W_i = I - \tfrac{1}{2}\{(I - \mathscr{P}_i{}^k)(I - \mathscr{P}_j{}^k) + (I - \mathscr{P}_i{}^k)(I - \mathscr{P}_l{}^k)\}$$

and

$$Q^* = I - \tfrac{1}{2}\{(I - \mathscr{P}_i{}^k)(I - \mathscr{P}_j{}^k) + (I - \mathscr{P}_i{}^k)(I - \mathscr{P}_l{}^k) + (I - \mathscr{P}_j{}^k)(I - \mathscr{P}_l{}^k) - (I - \mathscr{P}_i{}^k)(I - \mathscr{P}_j{}^k)(I - \mathscr{P}_l{}^k)\}.$$

Thus

$$W_i - Q^* = \tfrac{1}{2}\mathscr{P}_i{}^k(I - \mathscr{P}_j{}^k)(I - \mathscr{P}_l{}^k) \equiv 0$$

by Lemma 4.2.

Lemmas 4.2 and 4.3 immediately give

THEOREM 4.1. *Let $u \in C^{k-1}(\partial T_h)$ and satisfy* (4.8) *at each vertex. Then Q^*u, with Q^* defined by* (4.1), *interpolates u and its first $k - 1$ derivatives on the boundary of T_h.*

Remark. Note that Lemma 4.3 also shows that Q^* is equivalent to the operator $\tilde{Q}$ of Ref. 1, Theorem 4, defined by

$$\tilde{Q} = \tfrac{2}{3} \sum_{i=1}^{3} \mathscr{P}_i^{\,k} - \tfrac{1}{6} \sum_{i \neq j} \mathscr{P}_i^{\,k} \mathscr{P}_j^{\,k} = \tfrac{1}{6} \sum_{i \neq j} \mathscr{P}_i^{\,k} \oplus \mathscr{P}_j^{\,k}.$$

LEMMA 4.4. *Q^* is exact for all polynomials of degree $3k - 1$ or less.*

Proof. Q^* clearly is exact for all polynomials in $\mathscr{T}_{2k-1}$. We need now consider only elements $u \in \pi_{3k-1}$, the set of polynomials of degree $3k - 1$ or less, such that $\mathscr{L}^k u = 0$. Thus we first consider polynomials of the form $x^i y^j$, $i + j < 3k$, $k \leqslant i$, $j \leqslant 2k - 1$. Then $\mathscr{P}_1^{\,k} x^i y^j = \mathscr{P}_2^{\,k} x^i y^j = x^i y^j$, and $\mathscr{P}_3^{\,k} x^i y^j = \mathscr{L}^k x^i y^j = 0$. Thus $Q^* x^i y^j = x^i y^j$. Similarly Q^* is exact for all polynomials of the form $x^i(h - x - y)^j$ and $y^i(h - x - y)^j$, $i + j < 3k$, $k \leqslant i$, $j \leqslant 2k - 1$. These polynomials are linearly independent and total $l = 3k(k + 1)/2$. Since $3k^2 + l = 3k(3k + 1)/2$, the number of elements in π_{3k-1}, the proof is complete.

Since $Q^*[x^k y^k (h - x - y)^k] = 0$, Q^* does not reproduce all polynomials of degree $3k$. This indicates that $(I - Q^*)u$ is $O(h^{3k})$, a result which we now prove.

THEOREM 4.2. *Let $u \in B^r_{s,3k-s}(0, 0)$, $s = [3k/2]$. Then there exist constants C_1 and C_2 independent of u and h such that*

$$\begin{aligned} &\|(I - Q^*)\, u(x, y)\|_{L_r(T_h)} \\ &\quad \leqslant C_1 h^{3k+1/r} \Big[\sum_{i<s} \| u_{i,3k-i}(0, \tilde{y})\|_{L_r[0,h]} + \sum_{j<3k-s} \| u_{3k-j,j}(\tilde{x}, 0)\|_{L_r[0,h]} \Big] \\ &\quad\quad + C_2 h^{3k} \| u_{s,3k-s}(\tilde{x}, \tilde{y})\|_{L_r(T_h)} . \end{aligned} \tag{4.15}$$

Proof. First, it is easily shown that the kernels

$$K^{i,3k-i}(x, y; \tilde{y}) = (I - Q^*)_{(x,y)}[x^{(i)}(y - \tilde{y})_+^{(3k-i-1)}], \qquad i < s,$$

and

$$K^{3k-j,j}(x, y; \tilde{x}) = (I - Q^*)_{(x,y)}[y^{(j)}(x - \tilde{x})_+^{(3k-j-1)}], \qquad j < 3k - s,$$

can each be bounded on T_h by a constant times h^{3k-1}, and that the kernel

$$K^{s,3k-s}(x, y; \tilde{x}, \tilde{y}) = (I - Q^*)_{(x,y)}[(x - \tilde{x})_+^{(s-1)}(y - \tilde{y})_+^{(3k-s-1)}]$$

can be bounded on T_h by a constant times h^{3k-2}. Thus

$$|(I - Q^*)\, u(x, y)| \leqslant \sum_{i<s} \tilde{C}_{i,3k-i} h^{3k-1+1/r'} \| u_{i,3k-i}(0, \tilde{y})\|_{L_r[0,h]} + \sum_{j<3k-s} \tilde{C}_{3k-j,j} h^{3k-1+1/r'} \| u_{3k-j,j}(\tilde{x}, 0)\|_{L_r[0,h]} + \tilde{C}_{s,3k-s} h^{3k-2+2/r'} \| u_{s,3k-s}(\tilde{x}, \tilde{y})\|_{L_r(T_h)} \tag{4.16}$$

where $1/r + 1/r' = 1$. Taking the L_r norm of both sides of (4.16) together with the triangle inequality gives (4.15) and proves the theorem.

We remark that a careful estimation of these kernels leads to explicit bounds, as in the preceding section.

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